Journal of Computational Physics 229 (2010) 6548–6557

Contents lists available at ScienceDirect

Journal of Computational Physics

journal homepage: www.elsevier.com/locate/jcp

Sinc collocation approximation of non-smooth solution of a nonlinear weakly singular Volterra integral equation

Gholam-Ali Zakeri *, Mitra Navab

Department of Mathematics, California State University – Northridge, Northridge, CA 91330-8313, USA

ARTICLE INFO

Article history:
Received 5 November 2009
Received in revised form 11 May 2010
Accepted 12 May 2010
Available online 21 May 2010

Keywords:
Nonlinear singular Volterra integral equation of second kind
Sinc collocation method
Weakly singular kernel
Whittaker's cardinal function

ABSTRACT

A numerical method based on sinc collocation approximation for a class of nonlinear weakly singular Volterra integral equations of a second kind with non-smooth solution is given. The numerical method given here combines a sinc collocation method with an explicit iterative process that involves solving a nonlinear system of equations. We provide an error analysis for the method. It is shown that the approximate solution converges to the exact solution at the rate of $\sqrt{M}\exp(-c\sqrt{M})$, where M is the number of collocation points and c is some positive constant. Some numerical results for several test functions are given to confirm the accuracy and the ease of implementation of the method.

1. Introduction

Many physical, chemical, and biological problems are modeled as nonlinear Volterra integral equations, such as reaction–diffusion problems, crystal growth, models describing the propagation of a flame (see e.g. [23,10] and especially [14] for many physical and engineering applications), mathematical models describing the behavior of viscoelastic materials in mechanics, superfluidity problems, and some newer applications on the study of soft tissues like mitral valves of the aorta in human heart (see [9] and the references therein).

This work is concerned with study of the numerical analysis of a class of nonlinear Volterra integral equation of a second kind which has a weakly singular kernel of the form

$$u(x) = f(x) + \int_a^x \frac{K(x,t)}{(x-t)^\alpha} u^p(t)\, dt, \tag{1}$$

where $a \leqslant x, t \leqslant b$, $p > 1$, and $0 < \alpha < 1$. Eq. (1) can arise in connection with some heat conduction problems with various class of mixed-type boundary conditions. For example, Lighthill [15] was among the pioneers to derive an integral equation that can be transformed into the above equation which describes the temperature distribution of the surface of a projectile moving through a laminar layer when $f(x) = 1$, $K(x,t) = (-\sqrt{3}/\pi)t^{1/3}$, $\alpha = 2/3$, $p = 4$ and $a = 0$, $b = 1$. Even for analytic functions $f(x)$ and $K(x,t)$, it is well known (e.g. see [5], and [1]) that derivative of the solution of the above equation, $u'(x)$ is singular at the left edge point of the interval of integration, $[a,x]$, and this is expected to cause a loss in global convergence of a collocation method. In the case of Eq. (1), $u'(x)$ behaves as $(x-a)^{-\alpha}$ as $x \to a^+$.

* Corresponding author. Tel.: +1 818 677 7816; fax: +1 818 677 3634.
E-mail addresses: ali.zakeri@csun.edu (G.-A. Zakeri), mitra.navab.46@csun.edu (M. Navab).

0021-9991/$ - see front matter
doi:10.1016/j.jcp.2010.05.010

Numerical approximations methods such as quadrature rules, finite differences, finite elements, and so on are generally use polynomials as basis functions to obtain approximate solutions that are sufficiently accurate in region where the function to be approximated is smooth (see e.g., [8,13]). However, such methods fail significantly in a neighborhood of singularities of the function. On the other hand, the numerical approximations obtained by using Whittaker's cardinal function yield much better results than those obtained by methods using polynomials in the case when singularities are present at an endpoint of the interval. These methods, however, may or may not yield better results in the absence of singularities. For a comprehensive study of numerical methods for Volterra integral equations we refer to [16,4,5], and the references therein, for single exponential sinc approximation methods to [17,26], and [27], for double exponential sinc transformation methods to [21,28] and [29].

In the present paper we develop a sinc collocation method for the nonlinear integral Eq. (1) that is based on the work of [24] for linear integral equation. [24] points out that "the extension of the method to nonlinear integral equations seems to be a more challenging task at this point." To our knowledge, no such extension is extant in the literature for the method given in [24]. Recently, [21] modified the method of [24] using a double exponential transformation for the linear case again and most recently [22] extending the method [24] to Fredholm case. In [24] the error analysis is based on an ambiguous limitation such as for all "*M* in a practical range," where *M* is the number of collocation points. In this paper, we set up the equations that gives the approximate solution for the integral equation in such a way that avoids such limitation. This is a sharp contrast between our approach and those in [24] and [21]. This paper is organized as follows. In Section 2 we present some definitions and preliminary results on sinc collocation method of single exponential function. Section 3 is devoted to a detailed derivation of our numerical algorithm, convergence and error analysis. Section 4 contains some numerical examples illustrating the applications of method described here that considers the rule of number of collocations. We end the paper with some closing remarks and conclusions.

2. Some preliminary results using sinc functions

In this section, we state some basic results about sinc function approximation. These important properties will enable us to solve the nonlinear singular Volterra integral equation. The basic sinc function is defined as

$$\operatorname{sinc}(x) = \begin{cases} \frac{\sin \pi x}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases} \tag{2}$$

Let j be an integer and h be a positive number. We define the jth translate of sinc function by

$$S(j,h)(x) \equiv \operatorname{sinc}(x/h - j) \tag{3}$$

for step size h, evaluated at x. Given a function f defined and bounded for all x in $(-\infty,\infty)$, the *Whittaker's cardinal function* of f is defined by

$$C(f,h)(x) = \sum_{j=-\infty}^{\infty} f(jh)S(j,h)(x). \tag{4}$$

Now, we want to extend the approximations on $\mathbb{R}$ to the finite interval (a,b). Since the integral equation is defined over a finite interval, and the sinc function maps $\mathbb{R}$ onto a finite interval, we need some transformation $\phi(x)$ that maps a finite interval (a,b) onto $\mathbb{R}$. Let

$$\phi(z) = \log\left(\frac{z-a}{b-z}\right), \tag{5}$$

be a conformal map which carries the eye-shaped complex domain

$$D = \left\{z : \left|\arg\left(\frac{z-a}{b-z}\right)\right| < d < \pi\right\} \tag{6}$$

onto the open infinite strip

$$D_d = \{z \in \mathbb{C} : |\mathrm{Im}(z)| < d < \pi\}. \tag{7}$$

Note that at $x = kh$ with k an integer, the translate of sinc reduces to the Kronecher delta, i.e., $S(j,h)(kh) = \operatorname{sinc}(k-j) = \delta_{kj}$. We define the basis functions on (a,b) by

$$S(j,h)(\phi(x)) = \operatorname{sinc}(\phi(x)/h - j). \tag{8}$$

Setting

$$\phi(x_k) = \log\left(\frac{x_k - a}{b - x_k}\right) = kh \tag{9}$$

we get

$$x_k = \phi^{-1}(kh) = \frac{a + be^{kh}}{1 + e^{kh}}. \tag{10}$$

Since $\phi(a) = -\infty$, and $\phi(b) = \infty$, then we have

$$S(j,h)(\phi(a)) = S(j,h)(\phi(b)) = 0. \tag{11}$$

Thus

$$S(j,h)(\phi(x_k)) = \mathrm{sinc}(\phi(x_k)/h - j) = \mathrm{sinc}(k-j) = \delta_{kj}. \tag{12}$$

Now, given a function f defined on (a,b), we define the truncated cardinal function by

$$C_{M,N}(f,h)(x) = \sum_{j=-M}^{N} f(x_j)S(j,h)(\phi(x)) \tag{13}$$

for some appropriate integers M and N to be determined later. This function agrees with f at each grid point $x_k = \phi^{-1}(kh)$, $k = -M,\ldots,N$, since we have

$$C_{M,N}(f,h)(x_k) = \sum_{j=-M}^{N} f(x_j)S(j,h)(\phi(x_k)) = \sum_{j=-M}^{N} f(x_j)\delta_{kj} = f(x_k). \tag{14}$$

For convenience, from hereafter we set the interval of integration to be $(0,l)$, thus

$$\phi(x) = \log\left(\frac{x}{l-x}\right), \tag{15}$$

and for the collocation process, we select the grid points so that $x_n = \phi^{-1}(nh) = lw_n$, with $0 \leqslant w_n \leqslant 1$, where

$$w_n = \frac{e^{nh}}{1+e^{nh}}. \tag{16}$$

Finally, we cite the following two results from [24] that their proofs and other similar results can be found in [17, Chapter 3].

Theorem 2.1 [24]. *Let the function f be analytic and bounded on domain D, where D is defined as in (6). Further, suppose there are positive constants β, γ, C_1 and C_1' such that*

$$|f(x)| \leqslant \begin{cases} C_1 x^{\beta}, & 0 < x \leqslant \frac{1}{2}l \\ C_1'(l-x)^{\gamma}, & \frac{1}{2}l < x < l \end{cases}. \tag{17}$$

Then, taking $h = \sqrt{\pi d/(\beta M)}$ and $N = [[\beta M/\gamma]] + 1$, we have

$$|f(x) - C_{M,N}(f,h)(x)| \leqslant C_2\sqrt{M}\exp\left(-\sqrt{\pi d\beta M}\right), \tag{18}$$

where the constant C_2 only depends on f, d, β and γ.

Theorem 2.2 [24]. *Let function f be analytic on domain D with*

$$\int_{\phi^{-1}(x+L)} |f(z)dz| \to 0 \text{ as } x \to \pm\infty, \tag{19}$$

where $L = \{Im\ (z): |Im\ (z)— < d\}$, D is defined as in (6), and

$$\lim_{\varphi\to\partial D}\ \inf_{\varphi\subseteq D} \int_{\varphi} |f(z)dz| < \infty. \tag{20}$$

Further, suppose there are positive constants β, γ, C_3 and C_3' such that

$$|f(x)| \leqslant \begin{cases} C_3 x^{\beta-1}, & 0 < x \leqslant \frac{1}{2}l \\ C_3'(l-x)^{\gamma-1}, & \frac{1}{2}l < x < l \end{cases}. \tag{21}$$

Then, taking $h' = \sqrt{2\pi d/(\gamma M')}$ and $N' = [[\gamma M'/\beta]] + 1$,

$$\left|\int_0^l f(x)dx - \frac{h'}{l}\sum_{j=-M'}^{N'} f(x_j)x_j(l-x_j)\right| \leqslant C_4\exp\left(-\sqrt{2\pi d\gamma M'}\right), \tag{22}$$

where the constant C_4 only depends on f, d, β and γ.

3. Approximation based on sinc collocation

Eq. (1) can be written as

$$u(x) = f(x) + T^{\alpha,p}u(x), \quad 0 \leqslant x \leqslant l,\ 0 < \alpha < 1,\ p > 1, \tag{23}$$

where $T^{\alpha,p}:C[0,l] \to C[0,l]$ denotes the weakly singular nonlinear operator

$$T^{\alpha,p}u(x) = \int_0^x \frac{K(x,t)}{(x-t)^\alpha} u^p(t)dt, \quad 0 \leqslant x \leqslant l. \tag{24}$$

It is assumed that the functions $f(x)$ and $K(x,t)$ are smooth on their respective domains, $[0,l]$ and $\{(x,t):0 \leqslant t \leqslant x \leqslant l\}$. Then under these smoothness conditions, the classical results [6] guarantees the existence of a unique solution $u(x) \in C[0,l]$.

Now, we want to introduce a sinc collocation method for approximating the solution u of the above integral equation over the interval $[0,l]$. We assume that u is analytic and bounded on the domain D, but, in general, it has unbounded derivative at $x = 0$, i.e., $u'(x) = O(1/x^\lambda)$ as $x \to 0^+$ where $0 < \lambda < 1$. We define the function $U(x)$ corresponding to $u(x)$ by

$$U(x) = u(x) - \left[u(0) + \frac{u(l) - u(0)}{l}x\right] \tag{25}$$

that satisfies $U(0) = U(l) = 0$. Since we want that $U'(x)$ behaves as $x^{-\lambda}$ as $x \to 0^+$ and $U(x) \to 0$ as $x \to l^-$, we assume that

$$|U(x)| \leqslant \begin{cases} C_1 x^{1-\lambda}, & 0 < x \leqslant \frac{1}{2}l \\ C_1'(l-x), & \frac{1}{2}l < x < l \end{cases}. \tag{26}$$

Thus $U(x)$ satisfies the hypotheses of Theorem 2.1 with $\beta = 1 - \lambda$ and $\gamma = 1$. So, the truncated cardinal function for $u(x)$ from (25) using (13) can be written as

$$C_{M,N}(u,h)(x) = u(0) + \sum_{j=-M}^{N} U(x_j)S(j,h)(\phi(x)) + \frac{u(l) - u(0)}{l}x. \tag{27}$$

Since $U(x)$ decreases slower in the neighborhood of zero, the left end point of the interval, than the right end point at l, thus we must have $M \geqslant N$. Using (11) and (12), we note that (27) interpolates u at $0, l$ and each grid point x_j, for $j = -M,\ldots,N$. Taking the step size $h = \sqrt{\pi d/((1-\lambda)M)}$ and $N = [[(1-\lambda)M]] + 1$, we have

$$|u(x) - C_{M,N}(u,h)(x)| \leqslant C_5\sqrt{M}\exp(-\sqrt{\pi d(1-\lambda)M}), \tag{28}$$

where C_5 is a positive constant.

Eq. (27) and inequality (28) suggest that the exact solution of (23) might be well approximated by a trial solution of the form

$$u_{M,N}(x) = c_{-M-1} + \sum_{j=-M}^{N} c_j S(j,h)(\phi(x)) + \frac{c_{N+1}}{l}x. \tag{29}$$

Then we use the quadrature rule of Theorem 2.2 to obtain an approximation of $T^{\alpha,p}$ given by

$$T^{\alpha,p}_{N',M'}u(x) = x^{1-\alpha}h' \sum_{n=-N'}^{M'} K(x,xw_n)u^p(xw_n)w_n(1-w_n)^{1-\alpha}, \tag{30}$$

where w_n is given by (16). Expression (30) provides an accurate approximation of $T^{\alpha,p}u(x)$ when the product $K(x,\cdot)u^p$ is analytic and bounded on the domain D where D is defined as in (6). Indeed, taking $h' = \sqrt{2\pi d/(M'(1-\alpha))}$, $N' = [[(1-\alpha)M']] + 1$, and using Theorem 2.2, we get

$$|T^{\alpha,p}u(x) - T^{\alpha,p}_{M',N'}u(x)| \leqslant C_6\exp(-\sqrt{2\pi d(1-\alpha)M'}) \tag{31}$$

for all $x \in [0,l]$ and M' is not necessarily the same as M. This can be done if the functions $K(x,\cdot)$ are analytic on the domain D and are uniformly bounded on this domain for all $x \in [0,l]$. Thus our numerical approximate solution $u_{M,N}(x)$ must satisfy

$$u_{M,N}(x) = f(x) + T^{\alpha,p}_{N',M'}u_{M,N}(x), \tag{32}$$

where $u_{M,N}(x)$ is given by (29). To find the coefficients of $u_{M,N}(x)$ we evaluate (32) at the grid points x_n, $n = -M,\ldots,N+1$. First we note that (11), (23), (24) and (29) give

$$u_{M,N}(0) = c_{-M-1} = f(0), \tag{33}$$

and

$$u_{M,N}(x_i) = \begin{cases} f(0) + c_i + \frac{c_{N+1}}{l}x_i; & i = -M,\ldots,N, \\ f(0) + c_{N+1}; & i = N+1. \end{cases} \tag{34}$$

Then evaluating (32) at the grid points gives the following system of equations

$$u_{M,N}(x_i) = f(x_i) + T^{\alpha,p}_{N',M'} u_{M,N}(x_i); \; i = -M, \dots, N+1, \tag{35}$$

where from (30), we have

$$T^{\alpha,p}_{N',M'} u_{M,N}(x_i) = x_i^{1-\alpha} h' \sum_{n=-N'}^{M'} K(x_i, x_i w_n) u^p_{M,N}(x_i w_n) w_n (1-w_n)^{1-\alpha}. \tag{36}$$

The system of equations in (35) completely determines a nonlinear system of $M + N + 2$ equations to be solved for c_i and then the approximate solution of Volterra integral equation is given by Eq. (29).

3.1. Solving nonlinear system of equations in (35)

The system of Eq. (35) can be written as $F(\mathbf{c}) = 0$ where $F : \mathbb{R}^{M+N+2} \to \mathbb{R}^{M+N+2}$ and $\mathbf{c} \in \mathbb{R}^{M+N+2}$. Solutions of this system are closely associated with finding global minimum of $W(\mathbf{c}) = F^T(\mathbf{c})F(\mathbf{c})$. All practical approaches to solve such a nonlinear system are iterative and currently, there are much interests for finding a more efficient method for solving such nonlinear systems, e.g. see [2,7,11,18], and [19]. System of Eq. (35) can be solved by a quadratically convergent Newton's method [20] or by a superlinearly convergent quasi-Newton's type such as Broyden's method ([3,25]) or by a family of Jacobian-free techniques known as Newton–Krylov methods, e.g. see the comprehensive review article [12] and the references therein. The Jacobian of (35) must be non-singular for the Newton's method to converge. The Jacobian matrix for (35) can be written succinctly as

$$J = \begin{pmatrix} A(x_i) - I & \vdots & B \\ \cdots & \cdots & \cdots \\ 0 & \vdots & -1 \end{pmatrix}, \tag{37}$$

where $A(x_i) = \{a_{i,j}(x_i)\}$ is a square matrix of size $(M + N + 1)^2$, with

$$a_{i,j}(x_i) = x_i^{1-\alpha} h' p \sum_{n=-N'}^{M'} K(x_i, x_i w_n) \left(u^{(m)}_{M,N}(x_i w_n)\right)^{p-1} w_n^2 (1-w_n)^{1-\alpha} S(j,h)(\phi(x_i w_n)) \tag{38}$$

$B = (x_i/l)$ is a $M + N + 1$ column vector, and I is the identity matrix. So, J is non-singular if $A(x_i) - I$ is non-singular. That is, $A(x_i)$ is non-singular if 1 is not one of its eigenvalues. To assure J is a well behaved non-singular matrix and have a unique solution, we require that $0 < \det(A(x_i) - I) < 1$. This gives a bound on the free parameters, M, N, h. The Newton's iterative process is given by $\mathbf{c}^{(k+1)} = \mathbf{c}^{(k)} - J(\mathbf{c}^{(k)})^{-1} F(\mathbf{c}^{(k)})$. In practice one does not need to find inverse of J. To avoid calculating the Jacobian inverse, instead, we solve the matrix system $J(\mathbf{c}^{(k)})\mathbf{y}^{(k)} = F(\mathbf{c}^{(k)})$ for $\mathbf{y}^{(k)}$ and then the new iterate is calculated from $\mathbf{c}^{(k+1)} = \mathbf{c}^{(k)} - \mathbf{y}^{(k)}$. We can summarize the result as follows:

Proposition 1. *The standard Newton's method when applied on system of nonlinear equations in (35) leads to a unique solution if $0 < \det(A(x_i) - I) < 1$ where the matrix $A(x_i)$ is given in (38).*

The convergence requirement stated in the above proposition is not easy to implement. In addition, the above process involves solving a system of equations for each iteration, it is costly. To reduce the calculations costs, we propose that (32) be rewritten as the following explicit iterative scheme

$$u^{(m+1)}_{M,N}(x) = f(x) + T^{\alpha,p}_{N',M'} u^{(m)}_{M,N}(x) \tag{39}$$

with the initial starting function $u^{(0)}_{M,N}(x)$. To avoid guessing such an initial starting function, we evaluate (39) at the grid points x_i which leads to the following set of nonlinear system of equations for c_j given by

$$u^{(m+1)}_{M,N}(x_i) = f(x_i) + x_i^{1-\alpha} h' \sum_{n=-N'}^{M'} K(x_i, x_i w_n) \left(u^{(m)}_{M,N}(x_i w_n)\right)^p w_n (1-w_n)^{1-\alpha}, \tag{40}$$

where

$$u^{(m)}_{M,N}(x_i w_n) = f(0) + \sum_{j=-M}^{N} c_j^{(m)} S(j,h)(\phi(x_i w_n)) + \frac{c^{(m)}_{N+1}}{l} x_i w_n \tag{41}$$

and

$$u^{(m+1)}_{M,N}(x_i) = \begin{cases} f(0) + c_i^{(m+1)} + \frac{c^{(m+1)}_{N+1}}{l} x_i, & i = -M, \dots, N, \\ f(0) + c^{(m+1)}_{N+1}, & i = N+1. \end{cases} \tag{42}$$

The above iterative system can be solved to obtain $c_j^{(m)}$'s when initial values $c_j^{(0)}$'s are selected properly. Rewriting the above system as an explicit iterative system for $c_j^{(m+1)}$'s, we get

$$(\mathbf{c}^{(m+1)})^T = (g_{-M}(\mathbf{c}^{(m)}), \ldots, g_{N+1}(\mathbf{c}^{(m)}))^T, \tag{43}$$

where $c_j^{(m+1)} = g_j(\mathbf{c}^{(m)})$ are given by

$$c_i^{(m+1)} = f(x_i) - f(0) - \frac{x_i}{l}(f(l) - f(0)) + h' x_i \sum_{n=-N'}^{M'} \left\{ x_i^{-\alpha} K(x_i, x_i w_n) \left(u_{M,N}^{(m)}(x_i w_n) \right)^p - l^{-\alpha} K(l, l w_n) \left(u_{M,N}^{(m)}(l w_n) \right)^p \right\} w_n (1 - w_n)^{1-\alpha} \tag{44}$$

for $i = -M, \ldots, N$ and

$$c_{N+1}^{(m+1)} = f(l) - f(0) + l^{1-\alpha} h' \sum_{n=-N'}^{M'} K(l, l w_n) \left(u_{M,N}^{(m)}(l w_n) \right)^p w_n (1 - w_n)^{1-\alpha}. \tag{45}$$

The Jacobian of (43) can be written as

$$G = \begin{pmatrix} A(x_i) - \frac{x_i}{l} A(x_{N+1}) & \vdots & X \\ \cdots & \cdots & \cdots \\ Y & \vdots & b_{N+1,N+1}(x_{N+1}) \end{pmatrix}, \tag{46}$$

where the components of matrix A are given in (38), $X = (b_{i,N+1}(x_i) - \frac{x_i}{l} b_{i,N+1}(x_{N+1}))$ is an $M + N + 1$ column vector and $Y = (a_{N+1,j}(x_{N+1}))$ is an $M + N + 1$ row vector. Here, the components of X are given by

$$b_{i,j}(x_i) = \frac{x_i^{2-\alpha} h' p}{l} \sum_{n=-N'}^{M'} K(x_i, x_i w_n) \left(u_{M,N}^{(m)}(x_i w_n) \right)^{p-1} w_n^2 (1 - w_n)^{1-\alpha}. \tag{47}$$

For uniqueness of the solution of (43) we must have $|G_{i,j}| \leqslant \frac{r}{M+N+1} < 1$, with $r < 1$ for each component of G. We note that for each $x \in [0, l]$

$$\begin{aligned} \|S(j,h)(\phi(x))\|_2^2 &= \sum_{j=-N}^{M} \frac{\sin^2(\pi(\phi(x)/h - j))}{(\pi(\phi(x)/h - j))^2} = \frac{\sin^2(\pi\phi(x)/h)}{\pi^2} \sum_{j=-N}^{M} \frac{1}{(\phi(x)/h - j)^2} \\ &\leqslant \frac{\sin^2(\pi\phi(x)/h)}{\pi^2} \left(\frac{N+M}{|(\phi(x)/h - M)(\phi(x)/h + N)|} + \frac{1}{(\phi(x)/h + N)^2} \right) \\ &\leqslant \frac{\sin^2(\pi\phi(x)/h)|}{\pi^2} \left(\frac{1}{N} + \frac{1}{M} + \frac{1}{N^2} \right). \end{aligned} \tag{48}$$

Analyzing the convergent requirement leads to the following result.

Proposition 2. *Let $(\mathbf{c}^{(m)})$ be the sequence given in (43). The sequence is convergent if*

$$\frac{l^{1-\alpha} h' p \tilde{k} \tilde{u}^{p-1} q}{\pi} \sqrt{\frac{1}{N} + \frac{1}{M} + \frac{1}{N^2}} < 1, \tag{49}$$

where $\tilde{k}, \tilde{u}$ represent the maximum values of absolute values of $K(l, l w_n), u_{M,N}^{(m)}(l w_n)$, respectively, $h' = \sqrt{2\pi d/(M'(1-\alpha))}$, $N' = [[(1-\alpha)M']] + 1, N = [[(1-\lambda)M]] + 1$ and

$$q = \sum_{n=-N'}^{M'} (w_n)^2 (1 - w_n)^{1-\alpha} \leqslant \left| \frac{1 - (2-\alpha) e^{nh'}}{(2-\alpha)(1-\alpha) h' (1 + e^{nh'})^{2-\alpha}} \right|_{-N'}^{M'} \Big|. \tag{50}$$

Inequality (49) provides a bound on M, and M' so that the sequence in (43) converges. Another restriction on M' is provided by

$$\left| \int_0^1 \frac{t\,dt}{(1-t)^\alpha} - h' \sum_{n=-N'}^{M'} w_n^2 (1 - w_n)^{1-\alpha} \right| \leqslant C_7 \exp\left(-\sqrt{2\pi d (1-\alpha) M'} \right), \tag{51}$$

where C_7 is a constant and the value of integral is $\frac{1}{(1-\alpha)(2-\alpha)}$. To characterize the order of convergence of iterates in (43) in terms of the rate of convergence of the relative residuals, we assume that the sequence $(\mathbf{c}^{(m)})$ converges to $\mathbf{c}^*$ as $m \to \infty$, where $\mathbf{c}^*$ is the solution of $\mathbf{c}^* = g(\mathbf{c}^*)$ given by (43). Then calculating $\|\mathbf{c}^{(m+1)} - \mathbf{c}^*\| = \|g(\mathbf{c}^{(m)}) - g(\mathbf{c}^*)\|$ by using (43) and expanding the terms in the right hand side using Taylor series expansions, we conclude with the following result.

Proposition 3. *Let $(\mathbf{c}^{(m)})$ be the sequence given in (43) which converges to $\mathbf{c}^*$ as $m \to \infty$, then*

$$\|\mathbf{c}^{(m+1)} - \mathbf{c}^*\| = O\left(\|\mathbf{c}^{(m)} - \mathbf{c}^*\|^p \right). \tag{52}$$

That is, $\mathbf{c}^{(m)} \to \mathbf{c}^*$ with strong order of convergence of at least p, $p > 1$.

3.2. Numerical algorithm

We select M' so that it satisfies inequality (51) and for the sake of simplicity we choose $M = M'$. Then from the relations given in Proposition 2, we calculate N, N', h and h'. Finally, we use inequality (49) to check the validity of the computed solution.

- Initialization steps.
  $\alpha = 1$, $k = 0$, $m = 0$, select ϵ and δ as tolerance, $c_i^{(0)} = 1$ for each i,$W(\mathbf{c}) = F^T(\mathbf{c})F(\mathbf{c})$.
- Coarse approximation steps.
  We use a steepest descent process to find a reasonably suitable initial approximation for $\mathbf{c}^{(0)}$ for the refinement steps. $\mathbf{c}^{(k+1)} = \mathbf{c}^{(k)} - \alpha\nabla W(\mathbf{c}^{(k)})$.
  If $W(\mathbf{c}^{(k+1)}) < \epsilon$ go to refinement steps, if not, check if $W(\mathbf{c}^{(k+1)}) < W(\mathbf{c}^{(k)})$ then set $\alpha = \alpha + 1$ and $\mathbf{c}^{(k)} = \mathbf{c}^{(k+1)}$, evaluate $\mathbf{c}^{(k+1)}$, otherwise set $\alpha' = \alpha$ and minimize a single variable function $T(\alpha) = \mathbf{c}^{(k)} - \alpha\nabla W(\mathbf{c}^{(k)})$ over interval $[\alpha' - 1, \alpha']$ then calculate $\mathbf{c}^{(k+1)}$.
- Refine approximation steps.
  Let $\mathbf{c}^{(0)} = \mathbf{c}^{(k+1)}$. Use the iterative system (43) to approximate $\mathbf{c}^{(m+1)}$. Stop when $\|\mathbf{c}^{(m+1)} - \mathbf{c}^{(m)}\| < \delta$.
- Approximate solution $u_{M,N}(x)$ using (29), and estimate the error.
- Check the validity of the solution.

3.3. Error analysis

Let $u(x)$ and $u_{M,N}$ be the exact solutions of (23) and (32), respectively. The error in the mth step of our successive approximation is given by $E_{M,N}^{(m)} = \sup_{0\leqslant x\leqslant l}|u(x) - u_{M,N}^{(m)}(x)|$. To get an upper bound on $E_{M,N}^{(m)}$, we note that

$$|u(x) - u_{M,N}^{(m)}(x)| \leqslant |u(x) - C_{M,N}(u,h)(x)| + |C_{M,N}(u,h)(x) - C_{M,N}(u_{M,N}^{(m)},h)(x)| + |C_{M,N}(u_{M,N}^{(m)},h)(x) - u_{M,N}^{(m)}(x)|. \tag{53}$$

Now, we want to get an upper bound on each term of the right hand side of (53). By Theorem 2.1, we have

$$\sup_{0\leqslant x\leqslant l} |u(x) - C_{M,N}(u,h)(x)| \leqslant C_8\sqrt{M}\exp\left(-\sqrt{\pi d(1-\lambda)M}\right) \tag{54}$$

and

$$\sup_{0\leqslant x\leqslant l} |C_{M,N}(u_{M,N}^{(m)},h)(x) - u_{M,N}^{(m)}(x)| \leqslant C_9(m)\sqrt{M}\exp\left(-\sqrt{\pi d(1-\lambda)M}\right) \tag{55}$$

for some positive constants C_8, and C_9, while the latter one depends on m. It remains to find an upper bound for the second term on the right hand side of (53). Using Schwarz inequality, we note that

$$\left|C_{M,N}(u,h)(x) - C_{M,N}(u_{M,N}^{(m)},h)(x)\right| = \sum_{j=-M}^{N}\left|\left(U(lw_j) - c_j^{(m-1)}\right)S(j,h)(\phi(x))\right| \leqslant \|U(lw_j) - c_j^{(m-1)}\|_2\|S(j,h)(\phi(x))\|_2. \tag{56}$$

Now we can consider $\psi = \sum_{j=-M}^{N}\xi_j c_j^{(m-1)}$ as a step function approximating $U(x)$ over the interval $[0,l]$ where ξ_j is the characteristic function over interval $[x_j,\, x_{j+1}]$ which assumes one on this interval and zero outside of that. The convergence of approximate solution to the exact solution guarantees the existence of a positive constant C_{10} depending on m such that

$$\|U(lw_j) - c_j^{(m-1)}\|_2 \leqslant \frac{C_{10}}{2^{M+N+1}}. \tag{57}$$

Using (48), (56), and (57), we get

$$\left|C_{M,N}(u,h)(x) - C_{M,N}(u_{M,N}^{(m)},h)(x)\right| \leqslant C_0(m)\sqrt{M^{-1}}2^{-M-N}. \tag{58}$$

Finally combining (53), (54), (55), and (58), we conclude

$$E_{M,N}^{(m)} \leqslant C(m)\sqrt{M}\exp(-c\sqrt{M}), \tag{59}$$

where $c = \sqrt{\pi d(1-\lambda)}$, and for some positive constant $C(m)$ depending on m. Choosing h relatively small and thus M relatively large can cause the error to be small. The preceding analysis can be summarized as the following result.

Theorem 3.1. *Let u be the solution of the weakly singular Volterra integral* Eq. (23). *Suppose u is analytic and bounded on the domain* (6). *Further, suppose the functions f(x), and K(x,·) of* (23) *are analytic on their corresponding domains and are uniformly bounded on these domains for all x ∈ [0, a]. Then* (29) *gives the approximate solution with error bound given by* (59) *when the step size is chosen so that the inequality* (49) *holds.*

4. Numerical results

Let us consider the following integral equations as our test problems. The exact solution of first three examples is $u(x) = \sqrt{x}$ and the exact solution of the last one is $u(x) = \sqrt[3]{x}$.

$$u(x) = -\frac{3\pi x^2}{8} + \sqrt{x} + \int_0^x \frac{1}{(x-t)^{1/2}} u^3(t)\,dt, \tag{60}$$

$$u(x) = \frac{\sqrt{x}}{15}(15 - 16x^2) + \int_0^x \frac{1}{(x-t)^{1/2}} u^4(t)\,dt, \tag{61}$$

$$u(x) = -\frac{5\pi x^3}{2} + \sqrt{x} + \int_0^x \frac{1}{(x-t)^{1/2}} u^5(t)\,dt, \tag{62}$$

$$u(x) = \frac{\sqrt[3]{x}}{15}(x - 15) + \int_0^x \frac{1}{(x-t)^{1/3}} u^3(t)\,dt. \tag{63}$$

The numerical experiments are implemented in C/C++. The programs are executed on a PC with 2.4 GHz Intel Core 2 Duo processor with 2 GB 667 MHz DDR2 SDRAM. The CPU times for the integral Eq. (61), were ranging from 67 s to 172 s depending on the size of M. For integral Eq. (61), we have used $\alpha = 1/2,\ p = 4,\ \lambda = 1/2,\ d = \pi/2,\ h' = \pi\sqrt{2/M'}$, $N' = [[M'/2]] + 1,\ h = \pi/\sqrt{M}$ and $N = [[M/2]] + 1$. So, it is sufficient to select the values of M and M'. In Tables 1 and 2, we list the maximum relative errors for several selected values of M and M', over the intervals $[0,1]$ and $[0,25]$. As it can be seen from these two tables, the method gives smaller relative error over smaller interval. This is expected since more function evaluation is taking place over interval $[0,l]$ and the round off error is playing a bigger role. Estimating the condition number for the system of equations in (35) is numerically very expansive, and for this reason it should be avoided as many authors do. Instead of estimating the condition number of system of Eq. (35), we use inequality (49) to check the validity of calculated solution. Our local estimation of condition number of (35), associated with the integral Eq. (61) is ranging from 45.14 to 57.42 over the interval $[0,1]$ at each collocation point for $M = 4$. These values are reasonably small and show the calculated solutions are reliable as it is evidenced from the maximum errors for each case. We get similar results for the other examples. That is, as l gets larger the method works better when number of collocations, M is chosen larger and increase the number of iterations to increase accuracy of the results. In Tables 3 and 4, we have presented the overall maximum errors for the integral Eqs. (60)–(63). We use the maximum number of iterations for Table 3 to be 20, 25, 30, 25, respectively, in Table 4, the number of iterations to 30, 35, 45, 30, respectively. As suggested by inequalities (49) and (51), to have the same level of accuracy, for bigger value of p, requires a larger value for M, the number of collocation points, to get the same order of accuracy. This is supported by numerical calculations as illustrated in Tables 3 and 4. Finally, as pointed out in Section 1, the main advantage of the sinc collocation method is that it gives a better result when derivative of the solution is singular at the left edge of interval of integration. All examples given in this section are supporting this property, i.e., derivative of their solutions behave as $x^{-\alpha}$ for some $\alpha > 0$. In addition the sinc method has an exponential rate of convergence which is achieved by using small number of collocation points as illustrated here.

Table 1
Maximum relative error estimates over the interval $[0,1]$, for $M, M' = 2, 4, 8$, where m represents the number of iterations. Estimates are obtained at $x = 0.2, 0.4, 0.5, 0.6, 0.8$, and 1.0.

m	Maximum relative error estimates for (61)		
	$M' = M = 2$	$M' = M = 4$	$M' = M = 8$
20	1.02E−2	1.92E−3	1.72E−4
40	3.25E−3	4.31E−4	3.04E−5
60	2.45E−4	6.01E−6	2.13E−6
80	4.34E−6	7.42E−7	1.42E−7

Table 2
Maximum relative error estimates over the interval $[0,25]$, for $M, M' = 2, 4, 8$, where m represents the number of iterations. Estimates are obtained at $x = 1, 5, 10, 15, 20$, and 25.

m	Maximum relative error estimates for (61)		
	$M' = M = 2$	$M' = M = 4$	$M' = M = 8$
30	3.01E−2	6.54E−3	1.12E−3
50	6.22E−3	6.51E−4	3.01E−4
70	1.13E−4	1.72E−5	3.03E−5
90	6.36E−5	7.14E−6	1.37E−6

Table 3
Maximum relative error estimates over the interval [0, 1], for M, M' = 2, 4, 8, the number of iterations are 20, 25, 30, 25, respectively. Estimates are obtained at x = 0.2, 0.4, 0.5, 0.6, 0.8, and 1.0.

Example	Maximum relative error estimates		
	$M' = M = 2$	$M' = M = 4$	$M' = M = 8$
1	1.02E−2	1.92E−3	1.71E−4
2	4.11E−2	3.52E−3	2.31E−4
3	5.32E−2	2.76E−3	5.12E−4
4	3.45E−2	3.71E−3	3.61E−4

Table 4
Maximum relative error estimates over the interval [0, 25], for M, M' = 2, 4, 8, the number of iterations are 30, 35, 45, 30, respectively. Estimates are obtained at x = 1, 5, 10, 15, 20, and 25.

Example	Maximum relative error estimates		
	$M' = M = 2$	$M' = M = 4$	$M' = M = 8$
1	3.01E−2	6.54E−3	1.12E−3
2	5.42E−2	4.23E−3	2.63E−3
3	6.11E−2	3.87E−3	4.39E−3
4	2.76E−2	5.89E−3	2.62E−3

4.1. Conclusions and closing remarks

This work is concerned with the extension of a collocation sinc approximation method to a class of nonlinear Volterra integral equation of the second kind that can arise in connection with many applications such as heat distribution problems. The method first introduced in [24], and extended for double exponential transformation in [21] and [22]. The original method of [24] had an ambiguous restriction on the number of collocation points which was later clarified for linear case in [21] and recently extended for linear Fredholm integral equations in [22]. We were able to rewrite a system of nonlinear equations in an explicit iterative form that it is easy to implement. We have provided the convergence and error analysis for the method. One of the possible extensions of the method given here is to use double exponential transformation as used by several authors recently and in connection with this is particular method for linear case by [21]. Another possible extension of the method with combination of double exponential transformations as base functions for collocation grid points is to approximate the derivative of a function to solve a class of nonlinear Volterra integro-differential equations. The values of sinc methods lie in the fact that they behave well and produces better results when solution function has singularities. Collocation methods provide a natural grading for selecting the nodes in the quadrature method that the unknown function needs to be evaluated.

Acknowledgments

The authors thank the three unknown referees for their helpful suggestions and comments.

References

[1] P. Baratella, A.P. Orsi, A new approach to the numerical solution of weakly singular Volterra integral equations, J. Comput. Appl. Math. 163 (2004) 401–418.
[2] M. Bierlaire, F. Crittin, M. Themans, A multi-iterate method to solve systems of nonlinear equations, Eur. J. Oper. Res. 183 (2007) 20–41.
[3] C.G. Broyden, A class of methods for solving nonlinear simultaneous equations, Math. Comput. 19 (1969) 577–593.
[4] H. Brunner, Collocation Methods for Volterra Integral and Related Functional Differential Equations, Cambridge University Press, 2004.
[5] H. Brunner, P.J. van der Houwen, The Numerical Solution of Volterra Equations, North-Holland, Amsterdam, 1986.
[6] T.A. Burton, Volterra Integral and Differential Equations, second ed., Elsevier, 2005.
[7] F.L. de Mendonce, R. Perez, V.L.R. Lopes, Inverse q-columns updating methods for solving nonlinear systems of equations, J. Comput. Appl. Math. 158 (2003) 317–337.
[8] T. Diago, P. Lima, M. Rebelo, Numerical solution of a nonlinear Abel type Volterra integral equation, Commun. Pure Appl. Anal. 5 (2006) 277–288.
[9] K. Dithelm, J.M. Ford, N.J. Ford, M. Weilbeer, Pitfalls in fast numerical solvers for fractional differential equations, J. Comput. Appl. Math. 186 (2006) 482–503.
[10] R. Gorenflo, S. Vessella, Abel Integral Equations–Analysis and Applications, Springer, Berlin, 1991.
[11] M. Heyouni, Newton generalized Hessenberg method for solving nonlinear systems of equations, Numer. Algorithms 21 (1999) 225–246.
[12] D.A. Knoll, D.E. Keyes, Jacobian-free Newton–Krylov methods: a survey of approaches and applications, J. Comput. Phys. 193 (2004) 357–397.
[13] P.K. Kythe, P. Puri, Computational Methods for Linear Integral Equations, Birkhäuser, Boston, MA, 2002.
[14] E.G. Ladopoulos, Singular Integral Equations–Linear and Non-Linear Theory and its Applications in Science and Engineering, Springer-Verlag, Berlin, Heidelberg, Germany, 2000.
[15] J.M. Lighthill, Contributions to the theory of the heat transfer through a laminar boundary layer, Proc. Roy. Soc. London 202A (1950) 359–377.
[16] P. Linz, Analytical and Numerical Methods for Volterra Equations, SIAM, Philadelphia, 1985.
[17] J. Lund, K.L. Bowers, Sinc Methods for Quadrature and Differential Equation, SIAM, Philadelphia, 1992.

[18] Y.-Z. Luo, G.-J. Tang, L.-N. Zhou, Hybrid approach for solving systems of nonlinear equations using chaos optimization and quasi-Newton method, Appl. Soft Comput. 8 (2008) 1068–1073.
[19] J.M. Martinez, Practical quasi-Newton methods for solving nonlinear systems, J. Comput. Appl. Math. 124 (2000) 97–121.
[20] J.J. Moré, M.Y. Cosnard, Numerical solution of nonlinear equations, ACM Trans. Math. Software 5 (1979) 64–85.
[21] M. Mori, A. Nurmuhammad, T. Murai, Numerical solution of Volterra integral equations with weakly singular kernel based on the DE-sinc method, Jpn. J. Ind. Appl. Math. 25 (2008) 165–183.
[22] T. Okayama, T. Matsuo, M. Sugihara, Sinc-collocation methods for weakly singular Fredholm integral equations of the second kind, J. Comput. Appl. Math. 234 (2010) 1211–1227.
[23] M. Rasty, M. Hadizadeh, A product integration approach based on new orthogonal polynomials for nonlinear weakly singular integral equations, Acta Appl. Math. 109 (2010) 861–873.
[24] B.V. Riley, The numerical solution of Volterra integral equations with nonsmooth solutions based on sinc approximation, Appl. Numer. Math. 9 (1992) 249–257.
[25] S. Schlenkrich, A. Walther, Global convergence of quasi-Newton methods based on adjoint Broyden updates, Appl. Numer. Math. 59 (2009) 1120–1136.
[26] F. Stenger, Numerical Methods Based on Sinc and Analytic Functions, Springer-Verlag, New York, 1993.
[27] F. Stenger, Summary of sinc numerical methods, J. Comput. Appl. Math. 121 (2000) 379–420.
[28] M. Sugihara, M. Takayasu, Recent developments of the sinc numerical methods, J. Comput. Appl. Math. 164-165 (2004) 673–689.
[29] K. Tanaka, M. Sugihara, K. Murota, Numerical indefinite integration by double exponential sinc method, Math. Comput. 74 (2005) 655–679.